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THE INVERSE PROBLEM OF THERMOELASTICITY OF OPTICAL TOMOGRAPHY†

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The determination of the stresses in a transparent body from the results of its transillumination [1] assuming weak optical anisotropy [2, 3] is considered. The side surface of the body is assumed to be free of loads. The phase difference and the isocline parameter [4] in a family of parallel planes are measured. From these investigations, from experimental data and the equations of equilibrium it is possible to determine only the component of the stress tensor σ_{zz} normal to the plane of transillumination [5, 6].

THE PROBLEM of finding residual components is connected with the inverse problem of the theory of elasticity, the solution of which has been considered previously [5, 7] for the case of stresses caused by external loads.

It is qualitatively an extremely complex problem to determine the internal stresses due to distortions. It is made somewhat easier for glass by the fact that in most cases the residual strains in it are spherical: $\epsilon^0_{xx}=\epsilon^0_{yy}=\epsilon^0_{zz}=\alpha T_0$ and can be characterized by a single parameter T_0, the effective temperature of the residual strains [8–10]. The solution of the inverse problem of thermoelasticity in the case of plane strain ($\sigma_{xz}=\sigma_{yz}=0$) enables the stress to be established completely [11]. Its particular solution for round specimens leads to the sum law [1, 10, 12], which was proposed in [13, 14] for finding the stress in the case of an arbitrary axisymmetrical distribution of the stress.

In this paper we formulate the boundary-value problem for determining the internal temperature stresses in the volume from the results of its continuous transillumination in a system of parallel planes. The problem does not have a unique solution for body shapes and stresses of general form and hence the stress can only be partially established from the solution of this problem. In particular, in solids of revolution with an axisymmetrical temperature distribution a plane stress state is possible, for whose determination it is necessary to employ additional transillumination in the meridian plane.

Note that in modern optical tomographs simultaneous transillumination is carried out using a wide beam, thereby enabling measurements to be made over a wide field [15].

1. In the case of weak optical anisotropy for transillumination in the x, y plane, one can measure two radiation integrals [2, 4] along the beam l

$$A(m,\theta)=\int(m_im_j\sigma_{ij}-\sigma_{zz})\,dl,\; H(m,\theta)=\int m_i\sigma_{iz}\,dl,\;\; i,j=x,y$$

Summation is carried out over repeated indices, m_i is the component of the unit vector normal to the beam l, $m_x=\cos\theta$, $m_y=\sin\theta$, and m is the distance from the origin of coordinates to the straight line l.

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The value of the axial component of the stresses σ_{zz} is found from the inversion of the linear combination of ray integrals [5, 7]

$$\int \sigma_{zz}\, dl = \frac{\partial}{\partial z} \int_{m}^{m_1} H(m', \theta, z)\, dm' - A(m, \theta, z) \tag{1.1}$$

Here m_1 is one of the extreme points of the projection of the contour of the cross-section on to the m axis. In addition, from the inversion of the ray integral

$$\int \frac{\partial}{\partial z} \sigma_{zz}\, dl = -\frac{\partial}{\partial m} H(m, \theta, z) + \sigma_{zz} \operatorname{ctg} \gamma \frac{d}{dm} \Gamma|_{l_0}^{l_1} \tag{1.2}$$

one can determine $\partial\sigma_{zz}/\partial z$. Here γ is the angle between the external normal n and the z axis, Γ is the length of the arc of the contour of the cross-section, l_0 and l_1 are the points of entry and exit of the beam, respectively, and the quantities $\sigma_{zz}(l_1)$, $\sigma_{zz}(l_0)$ at the ends of the beam of the convex contour are found by means of tangential transillumination at these points from the values of the ray integral $A(m, \theta)$ and the boundary conditions.

Hence, the use of the procedure of inversion of the Radon transformation (1.1) and (1.2) enables us to determine the value of the component σ_{zz} and its partial derivatives with respect to z in the specimen.

We will consider the problem of finding the residual components of the stress tensor for a distribution of σ_{zz} specified in the body assuming that the distortion is caused by the temperature, and we will therefore start from the Duhamel–Neumann relations [16]

$$\sigma_{ij} = 2\mu\epsilon_{ij} + (\lambda\epsilon_{kk} - \varphi)\delta_{ij};\ i, j = x, y, z$$

Here $\varphi = 3K\alpha(T - T^0)$ takes into account the effect of the temperature on the stress and on the residual strains, i.e. T is the sum of the effective residual strain and the actual temperature of the specimen.

We will represent the required stresses in the form of the sum of solutions of the first and second kind (normal rotation) [17]

$$\begin{aligned}
&\sigma_{xx} = \frac{\partial^2}{\partial y^2}\Phi - \frac{\partial}{\partial z}\tau + 2\frac{\partial^2}{\partial x \partial y}N,\quad \sigma_{xy} = -\frac{\partial^2}{\partial x \partial y}\Phi - \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)N \\
&\sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial}{\partial z}\tau - 2\frac{\partial^2 N}{\partial x \partial y} \\
&\sigma_{xz} = \frac{\partial}{\partial x}\tau + \frac{\partial^2 N}{\partial z \partial y},\quad \sigma_{yz} = \frac{\partial}{\partial y}\tau - \frac{\partial^2 N}{\partial z \partial x}
\end{aligned} \tag{1.3}$$

The resolving functions τ, Φ, N are defined by the equations

$$\Delta_4 \tau = -\frac{\partial}{\partial z}\sigma_{zz} \tag{1.4}$$

$$\Delta_4 \Phi = \nu\sigma_{zz} + (1 - \nu)\frac{\partial}{\partial z}\tau + (1 - 2\nu)\varphi \tag{1.5}$$

$$2\mu(1 - \nu)\frac{\partial w}{\partial z} = (1 - 2\nu)(\sigma_{zz} - \varphi) - \nu\Delta_4\Phi,\quad \frac{\partial \Phi}{\partial z} = 2\tau - 2\mu w \tag{1.6}$$

$$\Delta N = 0 \tag{1.7}$$

((1.4) and (1.5) are the equations of equilibrium, and (1.6) are the Duhamel–Neumann relations).

Eliminating w and φ from (1.5) and (1.6) we obtain a system of two resolving equations for τ and Φ: (1.4) and

$$\Delta\Phi = \frac{\partial^2}{\partial z^2}\Phi + \Delta_+\Phi = \sigma_{zz} - \frac{\partial\tau}{\partial z} \tag{1.8}$$

which define the solutions of the first kind, and Laplace equations (1.7) for the potential of the normal rotation N. We note immediately that we have the following particular solution for Eq. (1.8)

$$\Phi_1(x, y, z) = \Phi_0(x, y, 0) - \int_0^z \tau(x, y, t)\, dt \tag{1.9}$$

which reduces it to a homogeneous equation. The function $\Phi_0(x, y, 0)$ is found by solving the two-dimensional problem

$$\Delta_+\Phi_0(x, y, 0) = \sigma_{zz}(x, y, 0) \tag{1.10}$$

In the case of plane strain, relation (1.10) expresses the well-known sum law and, together with the boundary conditions, completely defines the stresses [11].

The boundary-value problem with equations (1.4), (1.7) and (1.8) on a load-free surface is closed by three boundary conditions (n_i are the components of the vector of the external normal).

To conclude this general formulation of the problem we will consider the question of the uniqueness of its solution, i.e. the possibility that temperature stresses exist for which $\sigma_{zz} = 0$. The necessary condition for the temperature for such states

$$\Delta\varphi = \frac{(1-\nu)}{(1-2\nu)}\frac{\partial^3}{\partial z^3}\tau \tag{1.11}$$

is obtained by calculating the Laplace operator of both sides of Eq. (1.5), taking relation (1.8) into account. As follows from Eq. (1.5), τ is a harmonic function of the variables x and y. If τ depends only on z, it follows from (1.3) that relation (1.11) expresses the necessary condition for a plane-stress state to exist ($\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$). Condition (1.11) was obtained previously in [16] for the last form of stress distribution, and specific examples of a plane-stress state due to temperature for certain body shapes were also given. The possibility that such solutions exist for an arbitrary shape is quite a complex problem and has not been considered in the literature.

Hence, but virtue of the non-uniqueness of the solution of the problem formulated above, transillumination of the body in a system of parallel planes does not enable us to determine the stressed state completely.

2. We will illustrate the above-mentioned features of the inverse problem of thermoelasticity for optical tomography using the example of the determination of the axisymmetrical stresses in solids of revolution. Confining ourselves to solutions of the first kind ($N = 0$), we will write the stresses in a cylindrical system of coordinates

$$\sigma_{\rho\rho} = [\frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}]\,\Phi - \frac{\partial\tau}{\partial z},\quad \sigma_{\rho z} = \frac{\partial\tau}{\partial\rho}$$
$$\sigma_{\rho\varphi} = -\frac{\partial}{\partial\rho}\,[\frac{1}{\rho}\frac{\partial}{\partial\varphi}\Phi],\quad \sigma_{\varphi\varphi} = \frac{\partial^2\Phi}{\partial\rho^2} - \frac{\partial\tau}{\partial z},\quad \sigma_{\varphi z} = \frac{1}{\rho}\frac{\partial\tau}{\partial\varphi} \tag{2.1}$$

It follows from the condition for the problem to be axisymmetrical that τ and Φ are independent of the angle φ, while $\sigma_{\rho\varphi} = \sigma_{\varphi z} = 0$. The resolving equation can be simplified to

$$\frac{\partial^2}{\partial z^2}\,\Phi + (\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho})\,\Phi = \sigma_{zz} - \frac{\partial\tau}{\partial z} \tag{2.2}$$

Expressing the quantity $\rho\sigma_{\rho\rho}$ in terms of $\partial\Phi/\partial\rho$ and $\partial\tau/\partial z$, by differentiating (2.2) with respect to ρ we can formulate the boundary-value problem for the component $\sigma_{\rho\rho}$ as follows:

$$\frac{\partial}{\partial\rho}\,[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2\sigma_{\rho\rho})] + \frac{\partial^2\rho\sigma_{\rho\rho}}{\partial z^2} = \frac{\partial\sigma_{zz}}{\partial\rho} + \rho\frac{\partial^2\sigma_{zz}}{\partial z^2} - 3\frac{\partial\sigma_{\rho z}}{\partial z}$$
$$-\rho\frac{\partial^3}{\partial z^3}\,[\tau_1 + \tau_0(z)],\quad \tau_1 = \int\sigma_{\rho z}(t, z)\,dt \tag{2.3}$$

((τ_0/z) is the above-mentioned arbitrary function).

Hence, the problem of establishing $\sigma_{\rho\rho}$, $\sigma_{\varphi\varphi}$ in the volume from the known values of σ_{zz} and $\sigma_{z\rho}$ reduces to solving a boundary-value problem defined by partial differential equations (2.3) and the boundary conditions on the free side surface $\rho = R(z)$

$$\sigma_{\rho\rho} = [dz\,R(z)/dz]^2\,\sigma_{zz} \tag{2.4}$$

The quantities σ_{zz} and $\sigma_{z\rho}$ in this case are related by Eq. (1.4), while the component σ_{zz} satisfies the condition of statics, namely, conservation of the principal force vector over the cross-section

$$\int_0^{R(z)} \sigma_{zz}(\rho, z)\,\rho\,d\rho = 0$$

The components σ_{zz}, $\sigma_{\rho z}$ are established from the initial measurement data using inversion of the Abel integrals

$$2\int_m^R \frac{\sigma_{zz}(\rho)}{\sqrt{\rho^2 - m^2}}\,\rho d\rho = \frac{\partial}{\partial z}\int_m^R H(n, z)\,dn - A(m, z)$$
$$2m\int_m^R \frac{\sigma_{\rho z}(\rho)}{\sqrt{\rho^2 - m^2}}\,d\rho = H(m)$$

We will begin a general investigation of the boundary-value problem by considering a homogeneous equation, equating the right-hand side of (2.3) to zero. It is a standard equation of mathematical physics, obtained when solving Laplace's equation for the angular harmonic $\sigma(\rho, z)\cos\varphi = \rho\sigma_\rho(\rho, z)\cos\varphi$. Such an equation arises, in particular, when a solid of revolution is twisted around an axis. Various methods of solving it have been developed in detail in problems of the twisting of shafts of variable cross-section, and the stress concentration which arises in this has been analysed.

The presence on the right-hand side of Eq. (2.3) of an arbitrary function $\tau_0(z)$ confirms the fact that in solids of revolution a plane-stress state is possible which cannot be determined by the transillumination method considered. Without discussing all the various methods of solving the boundary-value problem (2.3) and (2.4) we will consider the method of matched asymptotic expansions. The simplest asymptotic solution is obtained for elongated parts of a body with a slowly varying shape. In this case, the principal term of the asymptotic expansion was found by solving the plane problem, i.e. on the left-hand side of Eq. (2.3) the derivative with respect to z is equated to zero. The solution of this equation can be expressed in the form of a sum

$$\sigma_{\rho\rho} = \sigma_0(z) + \sigma^*_{\rho\rho} \tag{2.5}$$

of the particular solution of Eq. (2.3)

$$\rho\sigma^*_{\rho\rho}(\rho, z) = -\int_0^z \sigma_{\rho z}(\rho, t)\, dt - \rho\frac{\partial}{\partial z}\tau_1 + \frac{1}{\rho}\int_0^\rho t\,\sigma_{zz}(t, 0)\, dt \tag{2.6}$$

of the slowly varying function $\sigma_0(z)$, found from the boundary condition (2.4) In view of the three-dimensional formulation of the problem, $\sigma_0(z)$ corresponds to the arbitrary function $\tau(z)$ in (2.3), while (2.5) gives one of the possible solutions of the three-dimensional problem.

Hence, the axisymmetrical stressed state in solids of revolution cannot be determined uniquely in the general case by transillumination in a system of planes orthogonal to the axis of revolution. The use of the sum law to determine the stresses is physically justified for sections having a smooth change in the surface and stresses along the axis of revolution, and corresponds to the zeroth approximation of the asymptotic solution. A more accurate asymptotic solution can be obtained using (2.5) and (2.6). If the above-mentioned conditions are not satisfied, to determine the stresses completely it is necessary to carry out an additional transillumination in the meridian plane. It can be shown that in this case the internal stresses are completely defined solely by the equations of equilibrium without having to make use of the Duhamel–Neumann relations.

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